HomeWork 1 : Applications of many-body Quantum Mechanics

Convention

If nothing is precised, I will used a, $a^\dagger$ for bosonic annihilition, creation operators, such that they fulfil the commutation relation $[a_\alpha, a^\dagger_\beta] = \delta_{\alpha\beta}$ (α, β being any quantum label of my states). And c, $c^\dagger$ will in principle refer to fermionic annihilation, creation operators, governed by an anticommutation relation $\{c_\alpha, c^\dagger_\beta\} = \delta_{\alpha\beta}$.

1 Playing with a and $a^\dagger$

1.1 Starting from the commutation relation for bosonic creation $a^\dagger$ and annihilation a operators.

$$[a, a^\dagger] = 1$$

show that

$$[a^\dagger a, a] = -a, \quad [a^\dagger a, a^\dagger] = a^\dagger$$

Using this result, show that, if $|\alpha\rangle$ represents an eigenstate of the operator $a^\dagger a$ with eigenvalue α, $a|\alpha\rangle$ is also an eigenstate with eigenvalue $\alpha - 1$ (unless $a|\alpha\rangle = 0$). Similarly, show that $a^\dagger|\alpha\rangle$ is an eigenstate with eigenvalue $\alpha + 1$.

1.2 If $|\alpha\rangle$ represents a normalized eigenstate of the operator $a^\dagger a$ with eigenvalue α for all $\alpha \geq 0$, show that

$$\begin{aligned} a|\alpha\rangle &= \sqrt{\alpha}\,|\alpha - 1\rangle \\ a^\dagger|\alpha\rangle &= \sqrt{\alpha + 1}\,|\alpha + 1\rangle \end{aligned}$$

Defining as the normalized vacuum $|\Omega\rangle$ the state annihilated by the operator a, show that $|n\rangle = (a^\dagger)^n|\Omega\rangle/\sqrt{n!}$ is a normalized eigenstate of $a^\dagger a$ with eigenvalue n.

1.3 Assuming that the operators a and $a^\dagger$ obey fermionic anticommutation relations, repeat 1.1 and 1.2

2 Interacting electron gas in "*second quantization*"

2.1 Show that the one-body kinetic energy operator takes the form

$$T = \int \mathrm{d}x \sum_\sigma c^\dagger_\sigma(x) \frac{\hat{p}^2}{2m} c_\sigma(x)$$

where the field operators obey the anticommutation relation $\{c_\sigma(x), c^\dagger_{\sigma'}(x')\} = \delta(x - x')\delta_{\sigma\sigma'}$ appropriate for the fermions. Hint : Remember that the kinetic energy operator is diagonal in the momentum space ...

2.2 Electrons interact via Coulomb potential, which is a two-body interaction :

$$V_{Coul} = \frac{1}{2} \sum_{i \neq j} \frac{q_e^2}{|\hat{x}_i - \hat{x}_j|},$$

where q_e is the electronic charge and $\hat{x}_i$ is the position operator of the i-th electron. Write V_{Coul} in second quantized form.

2.3 Changing to the basis in which the non-interacting Hamiltonian is diagonal, reexpress the Coulomb interaction. Show that the latter is non-diagonal and scatters electrons between different quasi-momentum states (see FIG. 1).

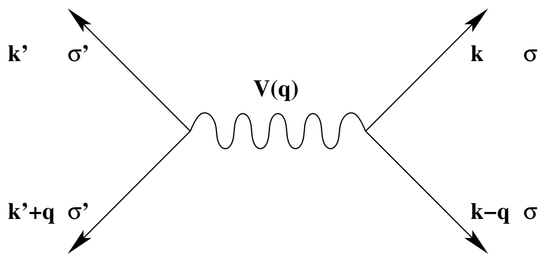

Figure 1: diagrammatic representation of the two-body Coulomb interaction

3 The cubic tight-binding model

The Hamiltonian of the tight-binding model on a $3D$ cubic lattice takes the form :

$$H_0 = -\sum_{<mn>} t_{mn} c^\dagger_{m\sigma} c_{n\sigma}$$

with ¡.¿ standing for summation other nearest neighbours and where σ is a spin index. The matrix elements t_{mn} are defined as followed :

$$t_{mn} = \begin{cases} t & m \text{ and } n \text{ nearest neighbors} \\ 0 & otherwise \end{cases}$$

Diagonalize H_0 by Fourier transform ($t > 0$) and comment on the form of the spectrum.

4 Non-interacting bosons

For independent harmonic oscillators (or non-interacting bosons) described by the Hamiltonian

$$H = \sum_i \varepsilon_i a^\dagger_i a_i$$

4.1 determine the equation of motion for the creation and annihilation operators in the Heisenberg representation,

$$a_i(t) = e^{\mathrm{i}Ht/\hbar} a_i e^{-\mathrm{i}Ht/\hbar}.$$

4.2 Give the solution of the equation of motion by (i) solving the corresponding initial value problem and (ii) by explicitly carrying out the commutator operations in the expression for $a_i(t)$.